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# CONDITIONS FOR PRODUCT FORM SOLUTIONS IN MULTIHOP PACKET RADIO NETWORK MODELS

José M. Brázio and Fouad A. Tobagi

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# Conditions For Product Form Solutions In Multihop Packet Radio Network Models*

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José M. Brázio and Fouad A. Tobagi
Computer Systems Laboratory
Department of Electrical Engineering
Stanford University, Stanford, CA 94305-2192



## Abstract

Consider multihop packet radio networks operating under a general class of channel access protocols. For the purpose of throughput analysis, analytical models are considered which describe the joint activity of the transmitters in the network, under the assumptions of heavy traffic and zero propagation and processing delays. The problem addressed in this report is that of finding conditions for the existence of product form solutions for the steady-state probabilities of these models. The main result states that a necessary and sufficient condition for a given network topology, channel access protocol, and traffic pattern, to lead to a product form solution is that the blocking between each pair of used links, as specified by the access protocol, be symmetric. This result assumes Poisson scheduling point processes associated with the links of the network. The proof is given in two steps: first, for systems where all packet length distributions are exponential, giving rise to Markovian processes; and second, for general packet length distributions (subject to the restriction of possessing a positive density almost everywhere), giving rise to Generalized Semi-Markov Processes. It is also shown that a product form solution does not exist whenever any of the scheduling point processes in the network is not Poisson. In addition, it is proven that the computation of the normalization factor appearing in the expression of the product form solution is an NP-hard problem. — R/H 7/



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# 1. INTRODUCTION

Packet radio networks have evolved over a time span of less than two decades from a fixed system for centralized communication, the ALOHA system [1], to mobile wide-area multihop systems, supporting store-and-forward operation, and possessing a distributed network control, allowing a variety of network services, and exhibiting a graceful degradation in the presence of malfunctions [2], [3]. Typically such systems operate using a single frequency, in a broadcast mode. Channel access schemes, as an essential component of a packet radio network, also accompanied this evolution. New channel access schemes with performance superior to that of the simple ALOHA protocol were developed, examples of which are the Carrier Sensing, Busy Tone, and Code Division Multiple Access families of access protocols [2], [3]. The need to assess the performance of these networks and protocols spurred the development of analytical models for the analysis of their throughput and delay characteristics [4]–[13]. Most early work [4]–[7] concerned either single hop fully-connected configurations, or multihop configurations with specific topologies and a limited number of hops. The study of these cases was usually done by formulating an appropriate Markovian model which was numerically solved for its steady-state probabilities, and from which performance measures such as throughput and delay were then derived. The inherent difficulty of the problem, however, made it hard to study networks with arbitrary topologies and a large number of hops. The difficulty stemmed from the fact that a packet radio network can be viewed as a queueing system in which the success of a message (and hence its service time) depends on the global state of the system. Such a system is difficult to analyze unless the problem possesses some special structure [8], [14].

More recently, analytical models were introduced for the study of general topologies and access protocols. Some of these models [9] achieve analytical tractability by assuming some form of decoupling between the activity of the different nodes. Other models, such as the one we consider in this Report, use assumptions of heavy traffic (thereby eliminating from consideration the queueing of packets at the different nodes), and lead to analytical solutions. These models are useful primarily for the purpose of determining the capacity of a packet radio network. The first such model was proposed by Boorstyn and Kershenbaum

for the analysis of multihop packet radio networks with zero propagation delay and exponential packet lengths, operating under Carrier Sense Multiple Access, and leading to a product form solution [10]. This model was extended in a number of subsequent works to encompass more general situations. In [11], still under the assumptions of exponential packet lengths, the model was extended to a number of other protocols. However, it was recognized that not all protocols lead to a product form solution. In [12] it was shown that, for CSMA networks with packet length distributions possessing a rational Laplace transform, the steady-state distribution depends on the packet length distributions only through their means, and thus, in what concerns the steady-state distribution, the results of [10] remain valid.

In this Report we address the problem of finding conditions for the existence of product form solutions. Section 2 describes the network model and its assumptions, and specifies the class of protocols encompassed by the analysis. Section 3 presents some preliminary definitions. Section 4 studies the existence of product form solutions in the case where all packet length distributions are exponential. In this section it is shown that the existence of a product form solution is equivalent to the reversibility of the underlying stochastic process, and it is given a simple characterization, in terms of the blocking between links (as specified by the channel access protocol), of the protocols which possess a product form solution. The results of this section had previously been reported in [13]. Section 5 treats the case of general packet length distributions. The main result of this section states that a product form solution exists in the general case if and only if a product form solution exists in the exponential case. Additionally it is shown that a product form solution does not exist whenever any of the link scheduling point processes is not Poisson. The Appendix gives the proof that the computation of the normalization factor appearing in the product form solution is an NP-hard problem.

## 2. GENERAL MODEL

We consider a packet radio network with $N$ nodes, numbered 1, 2, ..., $N$, which utilize a single broadcast radio channel. The topology of the network is given by a hearing matrix $\mathbf{H} = [h_{ij}]$, where

$$h_{ij} = \begin{cases} 1 & \text{if } j \text{ can hear } i \\ 0 & \text{otherwise.} \end{cases}$$

Thus each nonzero entry $h_{ij}$ in the hearing matrix corresponds to a directed radio link in the network from node $i$ to node $j$, and vice-versa. We call node $i$ the *source* and node $j$ the *destination* for that particular link. Due to the broadcast nature of the channel, a message sent over a given link will reach nodes other than its intended receiver, eventually colliding with messages destined to these nodes.

The traffic requirements for each link are assumed to be dictated by the end-to-end traffic requirements together with a static routing function. It may happen that for some links the required traffic is zero. We refer to these links as *unused links*, and all other links as *used links*. We say that a used link is *active* whenever a transmission is taking place over that link; i.e., whenever the source node is transmitting a message intended to the destination node on that link. Throughout this Report we consider all used links to be numbered $1, 2, \ldots, L$, and we let $\mathcal{L} \triangleq \{1, 2, \ldots, L\}$. For link $i \in \mathcal{L}$, we denote by $s_i$ its source node, and by $d_i$ its destination node. Alternatively we represent link $i$ by the ordered pair $(s_i, d_i)$.

The dynamics of link activity in the network is conditioned by the channel access protocol in use. An access protocol is a set of rules which, given the current global state of the network, determines whether or not an inactive link can become active. For most protocols of interest, a sufficient description of the network state is one which includes the information, for each node, as to whether the node is idle, transmitting a packet, or receiving a packet, and in the last two cases, the destination or source node, respectively. For practically realizable protocols, the rules embodied in the access protocol are constrained to be defined only in terms of information that can be made available

locally at the source node of the link, such as the state of the receiver at that node, and the state of the transmitters in some neighborhood of it. *We will consider here only the class of protocols for which the decision on whether or not to transmit can be expressed as a function of only the set of active links.* For this class of protocols, and in what concerns the activity of the transmitters, the activity of the receivers can be ignored*. The above class of protocols possesses some desirable analytical properties which do not exist in other classes of protocols, while at the same time containing protocols of great importance for the applications, such as the nonpersistent Carrier Sensing Multiple Access protocol, and the Busy Tone Multiple Access family of protocols†.

Since the entire packet radio network operates using a single radio frequency, each node in the network has one transmitter, but can in general have more than one outgoing link. We consider that each outgoing link at a node has a separate queue for the packets to be transmitted on it, and that the transmitter is shared among all queues at that node. To avoid repeated interference between transmissions in the network, transmission requests for the various queues at a node are scheduled according to random point processes, one for each queue. In this study, we consider the point process for link $i \in \mathcal{L}$ to be Poisson with rate $\lambda_i$ ($\lambda_i > 0$), independent of all other such processes in the network.

Consider a point in time defined by the point process for some link $i$. If the queue is empty, this scheduling point is ignored. If the queue is nonempty then a packet in the queue is considered for transmission. The transmission may or may not take place depending on the status of the transmitter at the source node (busy or idle), the priority structure (if any) among the queues at the source node, the channel access protocol in use, and the current activity on the network. If the transmission is inhibited, or if the transmission is undertaken but unsuccessfully (due to a collision at the intended destination

*Note that an *implementation* of the protocol at a given node might use information provided by the receiver at that node in order to assess whether or not a given link is active; this is the case, for example, with CSMA in narrowband channels. Our restriction, however, requires that, given the *complete* state description of the network, the protocol bases its decisions on the set of active links only.

†An example of a protocol which does not belong to the class under consideration is furnished by *disciplined ALOHA* [2], [3], in which a node is allowed to transmit at any time except when it is already transmitting, or is receiving (i.e., locked onto) some packet destined to it.

or to a preemption by another transmission at the source), then the packet in question (or any other packet in the queue, for that matter, depending on the queue discipline) is reconsidered at the next point in time. Otherwise (i.e., the transmission is successful), the packet is removed from the queue, and the same process is repeated at the next scheduling point for that link.

It is assumed in this study that at each scheduling point of the point process there is a packet in the queue for consideration (i.e., heavy traffic is assumed). It is also assumed that neither preemption, priority functions, or collision detection are supported at the nodes. In addition, we assume the transmission time of the messages transmitted over link $i$ to have a distribution function $B_i(\cdot)$ with mean $\mu_i^{-1} < \infty$, and to be redrawn independently from this distribution each time the message is transmitted. The functions $B_i(\cdot)$ are assumed to possess a positive density almost everywhere. We also assume infinite buffer space for each queue, and instantaneous and perfect acknowledgments for each link, providing immediate feedback regarding the success or failure of each transmission.

This Report is devoted to the study of the conditions under which the process $\{X(t) : t \geq 0\}$, where $X(t)$ is defined as the set of the links which are active at time $t$, has a product form stationary distribution. The stationary distribution of this process is important in that, from it, network performance measures, in particular throughput, can be derived. These applications will be considered in a separate paper.

In the case where all $B_i(\cdot)$, $i \in \mathcal{L}$, are exponential (i.e., the period of time that a link remains active is exponentially distributed), and given that the scheduling point processes which determine the points in time at which links can become active (as determined by the access protocol) are Poisson, $\{X(t) : t \geq 0\}$ is a continuous time Markov chain. The precise formulation of this Markov chain as well as the conditions under which it possesses a product form solution are presented in Section 4. When some of the $B_i(\cdot)$ are nonexponential, $\{X(t) : t \geq 0\}$ is no longer Markovian. However, it will be shown in Section 5 that this process has the structure of a general class known as Generalized Semi-Markov Processes (GSMPs). By exploiting the general properties of these processes we shall be able to derive the conditions under which a product form solution exists.

## 3. PRELIMINARY DEFINITIONS

Given an access protocol, we say that link $i \in \mathcal{L}$ blocks link $j$ if, whenever link $i$ is active, the protocol used does not allow a scheduling point for link $j$ to result in an actual transmission. It is to be noted that if link $i$ blocks link $j$, it does not necessarily follow that link $j$ blocks link $i$.

Let $D$ be a set of links in $\mathcal{L}$. We say that $D$ blocks link $j \in \mathcal{L} - D$ if there exists some link $i \in D$ which blocks $j$. We define $U(D)$ to be the set of all links in $\mathcal{L} - D$ which are not blocked by $D$.

In later treatments, the following two protocols are used as examples:

(i) Nonpersistent Carrier Sense Multiple Access (CSMA)[7]: under CSMA, a link will be blocked whenever its source node detects a transmission by any other source node that it can hear; i.e., link $(s_j, d_j)$ is blocked by $(s_i, d_i)$ whenever $h_{s_i s_j} = 1$, or $s_i = s_j$;

(ii) Idealistic Busy Tone Multiple Access (I-BTMA)[15]: this protocol assumes the existence of a separate channel for a busy tone. The destination of a link emits a busy tone whenever that link is active. A link is blocked if its source node hears either a transmission or a busy tone; i.e., link $(s_j, d_j)$ is blocked by $(s_i, d_i)$ if either $h_{s_i s_j} = 1$, $h_{d_i s_j} = 1$, or $s_i = s_j$.

## 4. EXPONENTIAL PACKET LENGTHS

### 4.1. Markovian Description of Network Activity

#### 4.1.1 State Space

As stated in section 2, when all packet length distributions are exponential, $\{X(t) : t \geq 0\}$ is a Markov chain. We now define the state space $S$ for this Markov chain. Since $X(t)$ is the set of all links that are active at time $t$, $S \subseteq 2^{\mathcal{L}}$. Given an access protocol and its blocking properties, not all subsets of $\mathcal{L}$ may be in $S$.

**Definition 4.1.1** $S$ is the collection of subsets of $\mathcal{L}$ that the system can reach starting from the idle state $\phi$ (i.e., all links inactive) by any sequence of *link activations* and *deactivations.*

**Definition 4.1.2** A subset $D = \{l_1, l_2, \ldots, l_n\}$ of $\mathcal{L}$ is said to be *directly reachable* if

there exists some permutation $(l_{i_1}, l_{i_2}, \ldots, l_{i_n})$ of $D$ such that $l_{i_j}$ is not blocked by $(l_{i_1}, l_{i_2}, \ldots, l_{i_{j-1}})$, $j = 2, \ldots, n$. That is, $D$ is directly reachable if it can be reached by *only* activating the links in it, in some order, starting from the idle state $\phi$.

**Lemma 4.1.3** If a subset $D = \{l_1, l_2, \ldots, l_n\}$ is directly reachable, then any subset $D' \subseteq D$ is also directly reachable.

**Proof:** Let $(l_{i_1}, l_{i_2}, \ldots, l_{i_n})$ be an ordered sequence of activations which allows $D$ to be reached. The ordered subsequence in $(l_{i_1}, l_{i_2}, \ldots, l_{i_n})$ corresponding to links in $D'$ is a sequence of activations which allows $D'$ to be reached directly. ∎

**Proposition 4.1.4** The state space $\mathcal{S}$ consists of $\phi$ and all subsets $D \subseteq \mathcal{L}$ that are directly reachable.

**Proof:** Clearly a set $D$ which is directly reachable belongs to $\mathcal{S}$. To prove the converse, we let $D \in \mathcal{S}$ be some subset that is reached via some sequence of states $D_0, D_1, \ldots, D_m$, with $D_0 = \phi$ and $D_m = D$, due to link activations and deactivations. (Note that since the process $\{X(t) : t \geq 0\}$ is such that no two events can occur at the same instant, then $|D_k| = |D_{k-1}| \pm 1$ for all $k = 1, 2, \ldots, m$). Since the first transition out of $D_0 = \phi$ must be an activation, there is some index $r \leq m$ such that $D_r$ is reached directly. Consider $D_{r+1}$. If $D_{r+1} = D_r \bigcup \{i\}$ for some $i$, then $D_{r+1}$ is clearly directly reachable. If $D_{r+1} = D_r - \{j\}$ for some $i$, then $D_{r+1}$ is also directly reachable, by Lemma 4.1.3. Applying the same argument to the remaining steps, we guarantee that $D$ is directly reachable.

∎

According to Proposition 4.1.4, one can generate the state space by the following algorithm, which is not necessarily claimed to be the most efficient for this purpose:

**begin**
$\mathcal{S} := \{\phi\}$ ;
$\mathcal{L} := \{1, 2, \ldots, L\}$ ;
**for** $k := 0$ **to** $L - 1$ **do**
**for every** $D \in \mathcal{S}$ **such that** $|D| = k$ **do**

**for every $l \in \mathcal{L} - D$ do**

**if $l$ is not blocked by $D$, then add $D \cup \{l\}$ to $\mathcal{S}$;**

**end.**

Throughout this Report we assume a fixed ordering of the state space $\mathcal{S}$, according to which the rows and columns of all the vectors and matrices to be considered are indexed.

**Remark 4.1.5** Given an access protocol and some state $D \in \mathcal{S}$, it should be noted that not all sequences of activations of its elements will necessarily allow $D$ to be reached from $\phi$. For example, consider the 4-node chain of Figure 1 with nonzero traffic requirement over links 1 and 5 only, and the I-BTMA access protocol. State $\{1, 5\}$ is an example of a state for which the order of activation is relevant. This state is reachable by the permutation $(1, 5)$, but not by the permutation $(5, 1)$.

**Remark 4.1.6** Recall that $\mathcal{L}$ is the set of all used links and thus $\lambda_i > 0$ for all $i \in \mathcal{L}$. Accordingly every state can be reached from the empty state in a nonzero period of time with nonzero probability. Similarly, the empty state can be reached from any other state in a nonzero period of time with nonzero probability (since $\mu_i > 0$ for all $i \in \mathcal{L}$). It then follows that all states communicate and the resulting Markov chain is irreducible.

### 4.1.2 The Equilibrium Equations

As noted above, the Markov chain $\{X(t) : t \geq 0\}$ is irreducible. Since the state space is

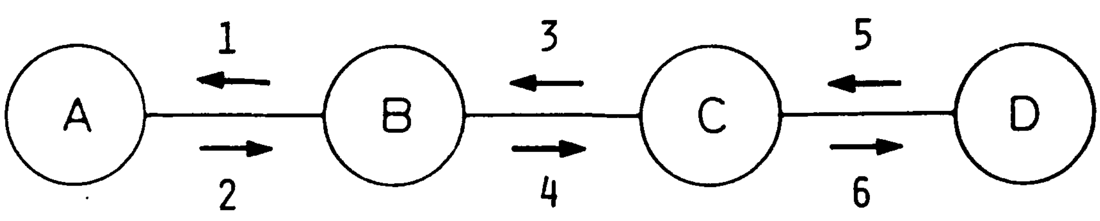


**Figure 1. A 4-node chain**

finite, the chain is then positive recurrent and ergodic. Thus the existence and uniqueness of a stationary distribution is ensured. We denote by $\{p(D) : D \in S\}$ the stationary probability distribution, and let $\mathbf{p} = \left(p(D)\right)_{D\in S}$ be the row vector of the steady-state probabilities.

Let the state of the system at time $t$ be $D \in S$, let $i$ be any link not blocked by $D$, and let $j \in D$. Given the assumptions in Section 2, the time to the next scheduling point of $i$ is exponentially distributed with parameter $\lambda_i$, and the time to the end of the transmission over link $j$ is also exponentially distributed with parameter $\mu_j$. Given that $X(t) = D$, the state of the system at time $t + \Delta t$ is given by (recall the definition of $U(D)$ in Section 3)

$$X(t+\Delta t) = \begin{cases} D\bigcup\{i\}, & i \in U(D), \text{with probability } \lambda_i \Delta t + o(\Delta t) \\ D - \{j\}, & j \in D, \text{with probability } \mu_j \Delta t + o(\Delta t) \\ D, & \text{with probability } 1 - (\sum_{i\in U(D)} \lambda_i + \sum_{j\in D} \mu_j)\,\Delta t + o(\Delta t) \end{cases}$$

This equation defines the transition rates which we need for writing the equilibrium equations [16]. Before doing so we have to introduce some further notation. For each $D \in S$, let $M(D)$ be the set of all links $i \notin D$ such that $D\bigcup\{i\} \in S$. Clearly $M(D) \supseteq U(D)$. Note however that it is not necessarily true that $M(D) = U(D)$. (See Example 4.1.7 below.) Let $J(D)$ to be the set of all links $j \in D$ such that $j$ is not blocked by $D - \{j\}$, i.e., such that $j \in U(D - \{j\})$. Clearly, $J(D) \subseteq D$. Here too, in general we have $J(D) \neq D$, as is also illustrated in Example 4.1.7. With these definitions, a sketch of the state-transition-rate diagram for state $D$ and the transitions to and from its neighbors can be seen in Figure 2. An equivalent description is given by the transition-rate matrix $\mathbf{Q} = \left[q(D, D')\right]_{D,D'\in S}$, where

$$q(D, D') = \begin{cases} \lambda_i, & \text{if } D' = D\bigcup\{i\},\ i \in U(D) \\ \mu_j, & \text{if } D' = D - \{j\},\ j \in D \\ -\left(\sum_{i\in U(D)} \lambda_i + \sum_{j\in D} \mu_j\right), & \text{if } D' = D \\ 0, & \text{otherwise} \end{cases}$$

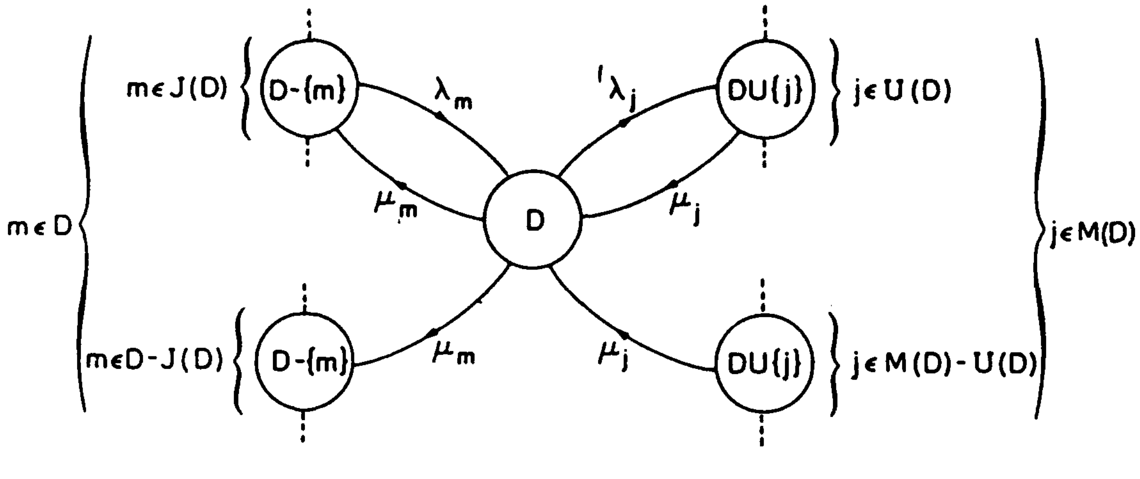


**Figure 2. Typical transitions to and from a state**

The equilibrium equations take then the form

$$p(D)\left[\sum_{i\in U(D)} \lambda_i + \sum_{j\in D} \mu_j\right] = \sum_{j\in J(D)} p(D-\{j\})\,\lambda_j + \sum_{i\in M(D)} p(D\cup\{i\})\,\mu_i, \quad D \in S \quad (4.1)$$

**Example 4.1.7** Consider the 4-node chain of Figure 1 with nonzero traffic requirement over links 1 and 5 only, and the I-BTMA protocol. The corresponding state-transition-rate diagram is shown in Figure 3. From the definitions we have that $J(\{1,5\}) = \{5\}$, $U(\{5\}) = \phi$, and $M(\{5\}) = \{1\}$. These are examples of states $D$ for which $M(D) \neq U(D)$, or $J(D) \neq D$.

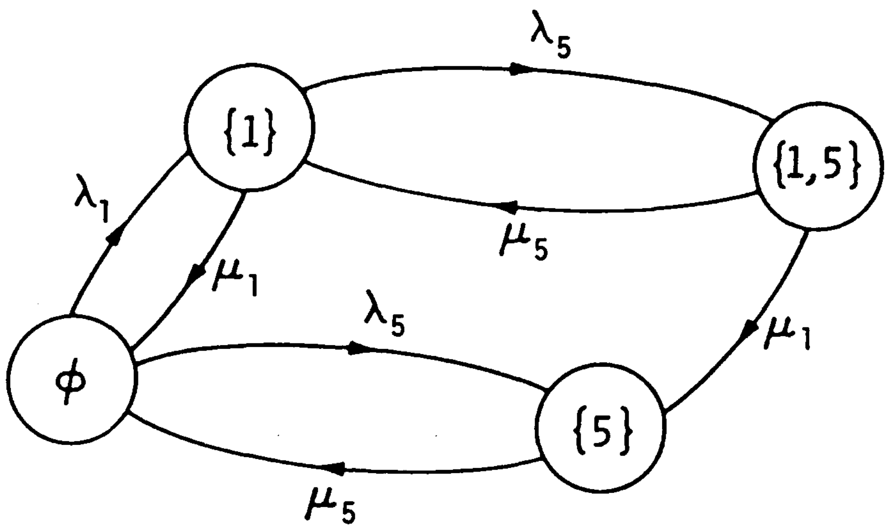


**Figure 3. State space for the Markov chain of Example 4.1.7**

## 4.2 Reversible Markov Chains and Product Form Solutions [17]

**Definition 4.2.1** A continuous time stochastic process $\{X(t)\}$ defined in $I = (-\infty, +\infty)$ is said to be *reversible* if for any $\tau \in I$, integer $n$, and $t_1 \leq t_2 \leq \cdots \leq t_n$ in $I$, $(X(t_1), X(t_2), \ldots, X(t_n))$ has the same distribution as $(X(\tau - t_1), X(\tau - t_2), \ldots, X(\tau - t_n))$.

For the particular case of Markov chains, reversibility has a simple characterization in terms of the transition rates and steady-state distribution, as given in the following proposition, whose proof can be found in [17].

**Proposition 4.2.2** A stationary continuous time Markov chain is reversible if and only if there exists a collection of positive numbers $\{\gamma(D) : D \in S\}$, summing to unity, such that

$$\gamma(D_1) \cdot q(D_1, D_2) = \gamma(D_2) \cdot q(D_2, D_1) \tag{4.2}$$

for all $D_1, D_2 \in S$, and where $q(D_i, D_j)$ is the rate of transitions from $D_i$ to $D_j$. When such a collection exists, it is the stationary probability distribution.

An equivalent necessary and sufficient condition for reversibility (called Kolmogorov's criterion) is that, for any finite sequence of states $D_1, D_2, \ldots, D_n \in S$, the transition rates satisfy

$$q(D_1, D_2)\, q(D_2, D_3) \cdots q(D_n, D_1) = q(D_1, D_n)\, q(D_n, D_{n-1}) \cdots q(D_2, D_1). \tag{4.3}$$

Suppose we are given a reversible Markov chain with state space $S$. Let $D_0$ be a fixed state and $D$ a generic state in $S$. Let $D_0, D_1, \ldots, D_m$ be any sequence of states in $S$, with $D_m = D$, such that between any two consecutive states of the sequence there exist nonzero transition rates. By repeated application of (4.2) it is easy to see that the steady-state probability distribution for such a Markov chain satisfies

$$p(D) = p(D_0) \prod_{k=1}^{m} \frac{q(D_{k-1}, D_k)}{q(D_k, D_{k-1})} \tag{4.4}$$

A solution with the form of (4.4) is called a *product form* solution. It is immediately seen that if the steady-state solution satisfies (4.4), then (4.2) is automatically satisfied for all $D_1, D_2 \in S$. Thus

**Proposition 4.2.3** A stationary continuous time Markov chain $\{X(t) : t \geq 0\}$ possesses a product form solution for the steady-state probability distribution if and only if it is reversible.

## 4.3 Criterion for the Existence of a Product Form

We use here the results of the previous section to determine the conditions on the access protocol, network topology, and traffic requirements under which the resulting Markov chain, defined in Section 4.1, is reversible and hence the global balance equations (4.1) have a product form solution.

**Lemma 4.3.1** $U(D) = M(D)$ for all $D \in S$ if and only if $J(D) = D$ for all $D \in S$.

**Proof:** We know already that $J(D) \subseteq D$ and $U(D) \subseteq M(D)$. To prove the desired equalities we only need to prove the reverse inclusions. Assume that $U(D') = M(D')$ for all $D' \in S$. It is evident that $J(\phi) = \phi$. Consider now any $D \in S, D \neq \phi$. For each $j \in D$, by definition $j \in M(D - \{j\})$. Since by hypothesis

$U(D-\{j\}) = M(D-\{j\})$, then $j \in U(D-\{j\})$. But this just means that $j \in J(D)$. Thus $D \subseteq J(D)$, for all $D \in S$. Conversely, assume that $D' = J(D')$ for all $D' \in S$. Call a state *maximal* if $M(D) = \phi$. Since $U(D) \subseteq M(D)$, for maximal states it is true that $U(D) = M(D)$. Let now $D \in S$ be a non-maximal state, and $j \in M(D)$. By hypothesis $J(D \cup \{j\}) = D \cup \{j\}$, which in particular implies that $j$ is not blocked by $D$, and thus that $j \in U(D)$. Hence $M(D) \subseteq U(D)$. ∎

**Proposition 4.3.2** The Markov chain $\{X(t) : t \geq 0\}$ is reversible if and only if

$$D = J(D) \tag{4.5}$$

for all $D \in S$ (or, equivalently, $U(D) = M(D)$ for all $D \in S$ ).

**Proof:** Assume that the Markov chain is reversible. Clearly (4.5) holds for $D = \phi$. Consider now $D \in S, D \neq \phi$, and $j \in D$. From (4.2) we have that

$$p(D) \cdot q(D, D-\{j\}) = p(D-\{j\}) \cdot q(D-\{j\}, D).$$

Since $q(D, D-\{j\}) = \mu_j > 0$ and $p(D) > 0$ for all $D \in S$, this last equation implies that $q(D-\{j\}, D) > 0$. But since $q(D-\{j\}, D)$ can only be either 0 (if $j \notin J(D)$) or $\lambda_j$ (if $j \in J(D)$), we necessarily conclude that $q(D-\{j\}, D) = \lambda_j$ and $j \in J(D)$. Then $D \subseteq J(D)$ for all $D \in S$, and consequently $D = J(D)$ for all $D \in S$. Conversely, assume that $J(D) = D$ for all $D \in S$. We now show that $\{\gamma(D) : \gamma(D) = \gamma_0 \prod_{i \in D} \frac{\lambda_i}{\mu_i}, D \in S\}$, with $\gamma_0$ chosen so that $\sum_{D \in S} \gamma(D) = 1$, is a collection of numbers that satisfies the conditions of Proposition 4.2.2. Let $D_1, D_2$ be any two states in $S$. Assume first that they are of either the form $D_1 = D$, $D_2 = D - \{j\}$, or the form $D_1 = D - \{j\}$, $D_2 = D$, for some $D \in S$ and $j \in D$. From the choice of $\gamma(D)$ we have

$$\gamma(D) = \frac{\lambda_j}{\mu_j} \gamma(D-\{j\}) \quad .$$

The transition rates between these two states are $q(D, D-\{j\}) = \mu_j$ and, from the assumption $J(D) = D$, $q(D-\{j\}, D) = \lambda_j$. Thus, in this case,

$$\gamma(D_1)q(D_1, D_2) = \gamma(D_2)q(D_2, D_1) \ .$$

For any other choice of $D_1$ and $D_2$, $q(D_1, D_2) = q(D_2, D_1) = 0$, and

$$\gamma(D_1)q(D_1, D_2) = \gamma(D_2)q(D_2, D_1)$$

is trivially verified. Thus (4.2) holds for all $D_1, D_2 \in S$, and $\{X(t) : t \geq 0\}$ is reversible, by Proposition 4.2.2. ∎

**Proposition 4.3.4** **(Criterion for the existence of a product form – exponential packet length distributions)** A necessary and sufficient condition for a channel access protocol, together with a given network topology and traffic requirements, in a system with exponential packet lengths, to have a product form solution is that, for all pairs of *used* links $i$ and $j$, link $j$ blocks link $i$ whenever link $i$ blocks link $j$.

**Proof:** We will prove the equivalence between the condition stated in the above criterion and the condition that $J(D) = D$, for all $D \in S$.

(a) $J(D) = D$, for all $D \in S$.
Assume that link $i$ blocks link $j$. If $j$ does not block $i$, we will have the situation depicted in Figure 4 in which $j \in \{i, j\}$ but $j \not\in J(\{i, j\})$, providing an instance of a state $D$ for which $J(D) \neq D$, which is a contradiction. Thus $j$ blocks $i$.

(b) There exists $D$ such that $J(D) \neq D$.
Since $J(D) \subseteq D$ and $J(D) \neq D$, there exists $j \in D$ such that $j$ is blocked by $D - \{j\}$. Let $i \in D - \{j\}$ be some link blocking $j$ and define $D' = \{i, j\}$. Since $D' \subseteq D$ then, by Lemma 4.1.3 and Proposition 4.1.4, $D' \in S$, and $D'$ can be directly reached by activating links $i$ and $j$ in some order. By hypothesis $i$ blocks $j$, and so $D'$ has to be directly reachable from $\{j\}$. Thus $j$ does not block $i$. ∎

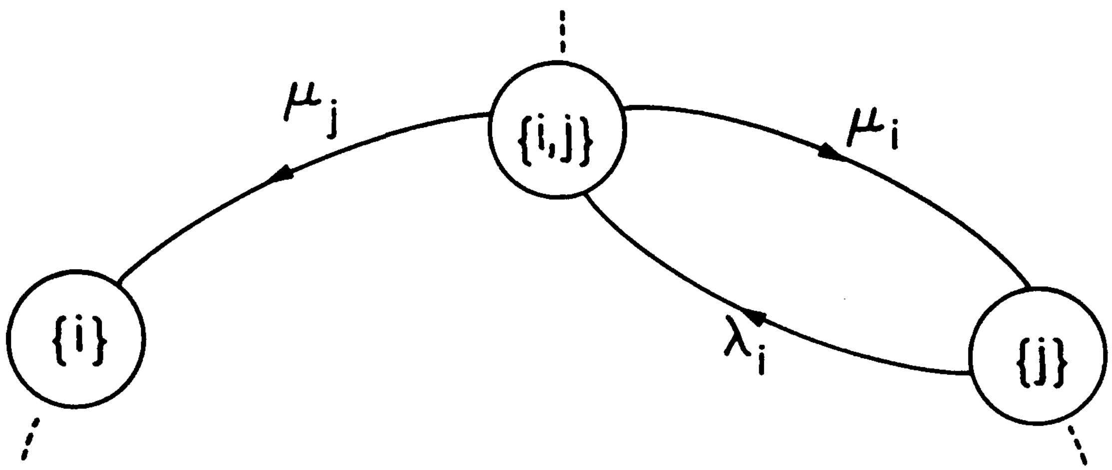


**Figure 4. Portion of a nonreversible chain**

Proposition 4.3.4 implies that, in a reversible chain and for any state $D \in S$, *any* order of activation of the links in $D$ allows $D$ to be reached directly from state $\phi$, and thus the situation depicted in Remark 4.1.5 does never occur.

For a reversible chain the stationary probability distribution is given by (4.4). From the particular form of the transition rates we have

$$p(D) = p(\phi) \prod_{i \in D} \frac{\lambda_i}{\mu_i} \tag{4.6}$$

for all $D \in S$. We can ask if there can exist protocols for which the corresponding Markov chain $\{X(t) : t \geq 0\}$ is not reversible, and yet the steady-state probabilities have the form (4.6).

**Proposition 4.3.5** (4.6) is a solution of the global balance equations (4.1) if and only if

$$D = J(D) \tag{4.5}$$

for all $D \in S$ (or, equivalently, $U(D) = M(D)$ for all $D \in S$).

**Proof:** Assume that $D = J(D)$ for all $D \in S$. By Proposition 4.3.2, $\{X(t) : t \geq 0\}$ is reversible and thus the steady-state probabilities have the form (4.6). Conversely, assume that (4.6) is a solution of (4.1). By substitution of (4.6) in (4.1) and simplification we obtain

$$\sum_{i \in M(D)-U(D)} \lambda_i = \sum_{j \in D-J(D)} \mu_j.$$

We now seek the conditions under which this equality can hold. Recall that a state $D$ of the Markov chain is said to be *maximal* if $M(D) = \phi$. Given a generic state $D$, define a *maximal path* starting at $D$ to be a finite sequence of states $D_0, D_1, \ldots, D_k$ such that $D_0 = D$, $D_{l+1} = D_l \cup \{i\}$ for some $i \in M(D_l)$, $l = 0, 1, \ldots, k-1$, and $D_k$ is a maximal state. Define the length of the maximal path to be $k$, and let $l(D)$ be the maximum of the lengths of the maximal paths starting at $D$. We shall now prove (4.5) by induction on $l(D)$. For $l(D) = 0$ we have that $D$ is a maximal state, for which $M(D) = U(D) = \phi$. Then

$$\sum_{j \in D-J(D)} \mu_j = 0.$$

Since, by assumption, $\mu_j > 0$, we obtain that $D = J(D)$. Assume now that, for $n$ a positive integer, (4.5) holds for all states $D'$ for which $l(D') \leq n$. Let $D$ be a state for which $l(D) = n + 1$. For all $j \in M(D)$, $D \cup \{j\}$ is a state for which $l(D) \leq n$. By the induction hypothesis we then have $J(D \cup \{j\}) = D \cup \{j\}$, which means in particular that $j$ is not blocked by $D$ or, in other words, that $j \in U(D)$. Then $U(D) = M(D)$ and

$$\sum_{j \in D-J(D)} \mu_j = 0.$$

Again, as all $\mu_j > 0$, it follows that $D = J(D)$. ∎

**Example 4.3.6** As an application of Proposition 4.3.4, we can now prove that, with a symmetric hearing matrix, nonpersistent CSMA always leads to a product form solution. Consider any two used links $i$ and $j$, and represent them as $(s_i, d_i)$ and $(s_j, d_j)$, respectively. Under CSMA, if $i$ blocks $j$, then either $s_i = s_j$ or $h_{s_i s_j} = 1$. The symmetry of the hearing matrix then implies that $j$ blocks $i$, and thus by Proposition 4.3.4 the stationary distribution will have a product form. If the hearing matrix is not symmetric we will not get a product form solution, except when all pairs of nodes $s_i$ and $s_j$ for which $h_{s_i s_j} = 1$ and $h_{s_j s_i} = 0$ are such that at least one element of the pair is the source of no used links.

**Example 4.3.7** The I-BTMA protocol will not, in general, lead to a product form solution. Indeed, if the network under consideration contains the subnetwork and traffic pattern of Example 4.1.7, we can find links $i$ and $j$ such that $i$ blocks $j$ but $j$ does not block $i$. For some specific topologies and traffic patterns, however, I-BTMA will have a product form solution. Examples of these are a star network with arms of length 1 and arbitrary traffic pattern, or a 4-node chain in which the outer nodes generate no traffic.

**Remark 4.3.8** In general, one may write the solution to (4.1) in the form

$$p(D) = p(\phi) f(D) \prod_{i \in D} \frac{\lambda_i}{\mu_i} \quad , \quad D \in S \tag{4.7}$$

where $f(\phi) = 1$ and $\{f(D) : D \in S, D \neq \phi\}$ is the solution of the following system of linear equations (obtained by substitution of (4.7) in (4.1) and simplification):

$$\left[ \sum_{i \in U(D)} \lambda_i + \sum_{j \in D} \mu_j \right] f(D) = \sum_{j \in J(D)} \mu_j f(D - \{j\})$$

$$+ \sum_{i \in M(D)} \lambda_i f(D \cup \{i\}), \quad D \in S \tag{4.8}$$

Indeed, this amounts to nothing else but a scaling of the unknowns in (4.1). The choice of the scaling given in (4.7) is remarkable in that the coefficients of the resulting set of equations (4.8) take on a very simple form. Furthermore, Proposition 4.3.5 implies that (4.8) has the solution $f(D) = \text{const.}, D \in S$, if and only if $J(D) = D, D \in S$. (Incidentally, in this case (4.8) coincides with the adjoint $\mathbf{Q f} = 0$ of the system of equations (4.1), where $\mathbf{f}$ is the column vector $\left(f(D)\right)_{D \in S}$.)

It is plausible that (4.8), together with (4.7), might present some computational advantages over (4.1) for the computation of the non-product form solutions. Indeed, one would intuitively expect that, for an "almost reversible" system (this notion being difficult to make precise) the solution of (4.8) would differ little from a constant, whereas (4.1) could give values of $p(D)$ for the various states $D$ widely different in magnitude, such as, for example, for states $D_1$ and $D_2$ of different cardinality. In this way, (4.8) would be expected to have a better numerical behavior. Along the same line of reasoning, the "deviation from constancy" of $\{f(D) : D \in S\}$ would be a measure of the "non-reversibility" of the Markov chain under consideration, and could be used to relate the performance of product form and non-product form protocols. Unfortunately, we were not successful at neither obtaining useful bounds for $f(D)$ nor, assuming that bounds were available, at relating those bounds to the differences in performance of product form and non-product form protocols.

# 5. GENERAL PACKET LENGTH DISTRIBUTIONS

## 5.1 Introduction

Let $X(t)$ be defined, as in section 2, as the set of links which are active at time $t$. We shall consider in this section the case where the lengths of the packets belonging to link $i \in \mathcal{L}$ have a distribution function $B_i(\cdot)$, with

$$\int_0^\infty x\, dB_i(x) = \mu_i^{-1} < \infty.$$

We shall only require that each $B_i(\cdot)$ possess a positive density almost everywhere. This condition will ensure the existence and uniqueness of a stationary distribution for $\{X(t) : t \geq 0\}$.

In this general case $\{X(t) : t \geq 0\}$ is no longer Markovian. However, it possesses the structure of a general class of processes known as GSMPs. We shall use general properties of these processes to obtain conditions under which $\{X(t) : t \geq 0\}$ has a product form steady-state distribution. The existence of a product form will be seen to be closely related to the insensitivity of the steady-state distribution with respect to the moments of second and higher orders of the distributions $B_i(\cdot)$, $i \in \mathcal{L}$.

### 5.2 Generalized Semi-Markov Processes* [18]–[21]

Consider a process $\{X^*(t) : t \geq 0\}$ which, at an arbitrary instant $t \geq 0$, can be in any one of the states $g$ of a finite state space $G$. Each state $g$ is itself a finite set of elements $s$ of a finite set $S$. (These elements can represent, for example, links in a packet radio network, or customers in a closed queueing system.) It is required that, for each $s \in S$, there exist at least one $g \in G$ such that $s \in g$. Suppose that $X^*(t) = g$. To each element $s \in g$ (which we shall say to be *active* at time $t$) there is associated a residual lifetime $Y_s(t) > 0$, determined as described in the following. We let $\mathbf{Y}(t) = (Y_s(t))_{s \in g}$ be the vector of the residual lifetimes of the elements active at time $t$. The lifetimes of the elements which are active at any given time decrease at unit rate†, $X^*(t)$ remaining in state $g$ as long all $Y_s(t)$ are positive. Eventually the lifetime of one of these elements will reach 0 (which we will refer to as the "death" of that element), at which time $X^*(t)$ jumps from state $g$ to a new state $g' \in G$. It is assumed that no two elements can die simultaneously. The state transitions are specified by a family of transitions probabilities

$$p = \{p(g, s, \cdot) : g \in G,\ s \in g\},$$

where $p(g, s, g')$ is the probability that the next state of $X^*(t)$ is $g' \in G$ given that the

*This section is a summary of the main definitions and results in [18] and [19]. We will try to conform to the notation of these papers whenever possible.

†See [20] for a generalization which allows a countable state space, and arbitrary rates of decrease for the residual lifetimes of the active elements. See also [21] for an alternative construction of a GSMP.

present state is $g \in G$ and that the state transition is caused by the death of $s \in g$. It is required that $p(g, s, g') = 0$ unless $g - \{s\} \subseteq g'$ (i.e., it is required that all other elements which are active in the old state remain active in the new state). Upon entering state $g'$ the residual lifetimes of the elements of $g'$ are determined as follows: the elements in $g - \{s\}$ keep the residual lifetimes they had at the time the state transition took place; a new element $s_i \in g' - (g - \{s\})$ is assigned a residual lifetime which is drawn, independently from the past, from a nonnegative distribution $\varphi_{s_i}$, with distribution function $F_{s_i}(\cdot)$, and mean $\eta_{s_i}^{-1} < \infty$. (These residual lifetimes continue to decrease at unit rate after the system enters state $g'$.) $X(0)$ is chosen to be some arbitrary state $g_0 \in G$, and the initial residual lifetime vector $\Upsilon(0)$ is obtained by assigning to the element $s \in g_0$ a residual lifetime $Y_s(0)$ drawn from the corresponding distribution $\varphi_s$. These distributions are assumed to be such that

**(C1)** no two deaths can occur simultaneously at any time, and

**(C2)** the resulting $X^*(t)$-process has a unique stationary distribution.

**Remark 5.2.1** In general, (C1) is easy to verify directly. A sufficient condition for (C2) is that all distributions involved possess a positive density almost everywhere [20]. In some cases it may be possible to obtain less restrictive conditions sufficient to ensure (C2).

**Definition 5.2.2** The collection $\Sigma = (G, S, p)$ is called a *generalized semi-Markov scheme (GSMS);* the process $\{X^*(t) : t \geq 0\}$ is called the *generalized semi-Markov process (GSMP)* based upon $\Sigma$ by means of the family $\{\varphi_s : s \in S\}$; the process $\{X^*(t), \Upsilon(t) : t \geq 0\}$ is called a *supplemented* GSMP.

**Definition 5.2.3** A GSMS is said to be *irreducible* if, for every pair $g, g' \in G$, there exist finite sequences $(g_0, \ldots, g_n), g_i \in G$, and $(s_1, \ldots, s_n), s_i \in S$, such that $p(g, s_0, g_1) \ldots p(g_n, s_n, g') > 0$.

**Definition 5.2.4** Let $\Sigma = (G, S, p)$ be an irreducible GSMS, and $\Phi(\Sigma)$ the collection of all families $\varphi = \{\varphi_s : s \in S\}$ of distributions concentrated on $(0, \infty)$ which imply the existence of a unique stationary distribution

for the corresponding supplemented GSMPs based upon $\Sigma$. Let $\Phi$ be a nonempty subset of $\Phi(\Sigma)$. $\Sigma$ is called *$\Phi$-insensitive* if every GSMP based upon $\Sigma$ by means of an element of $\Phi$ has the same stationary distribution.

**Notation** Let $S' \subseteq S$. We denote by $\Phi_{S'}(\eta_s : s \in S)$ the family of distributions

$$\{\varphi : \varphi \in \Phi(\Sigma), \varphi_s \quad E_{\eta_s}(\cdot) \text{ for } s \not\in S', \varphi_s \text{ arbitrary with mean } \eta_s^{-1} \text{ for } s \in S'\},$$

where $E_\eta(\cdot)$ represents an exponential distribution with parameter $\eta$. We also set $\Phi_{s_0}(\eta_s : s \in S) \triangleq \Phi_{\{s_0\}}(\eta_s : s \in S)$.

We now state the main results of interest for our applications. It is assumed throughout that the GSMS $\Sigma = (G, S, p)$ is irreducible.

**Proposition 5.2.5** An GSMS $\Sigma$ is $\Phi_{S'}(\eta_s : s \in S)$-insensitive if and only if $\Sigma$ is $\Phi_{s'}(\eta_s : s \in S)$-insensitive for every $s' \in S'$.

**Proposition 5.2.6** An GSMS $\Sigma$ is $\Phi_{S'}(\eta_s : s \in S)$-insensitive if and only if the stationary distribution of every supplemented GSMP based upon $\Sigma$ by means of a family $\varphi \in \Phi_{S'}$ is of the form

$$P\{X(t) = g, Y_s(t) \leq x_s, s \in g \cap S'\} = p_g \prod_{s \in g \cap S'} \eta_s \int_0^{x_s} (1 - F_s(t))\, dt,$$

where $\{p_g : g \in G\}$ is the steady-state probability distribution of the GSMP based upon $\Sigma$ by means of the exponential family in $\Phi_{S'}(\eta_s : s \in S)$.

**Remark 5.2.7** $\{p_g : g \in G\}$ is the normalized solution of the system of equations

$$p_g \sum_{s \in g} \eta_s = \sum_{g' \in G} p_{g'} \sum_{s \in g'} p(g', s, g)\, \eta_s, \quad g \in G. \tag{5.1}$$

These are the global balance equations for the GSMP based upon $\Sigma$ by means of the exponential family in $\Phi_{S'}(\eta_s : s \in S)$.

**Proposition 5.2.8** $\Sigma$ is $\Phi_{s_0}(\eta_s : s \in S)$-insensitive if and only if there exists a distribution $\{p_g : g \in G\}$ that satisfies (5.1) and

$$p_g \eta_{s_0} = \sum_{g' \notin G_0} p_{g'} \sum_{s \in g'} p(g', s, g)\, \eta_s + \sum_{g' \in G_0} p_{g'} p(g', s_0, g)\, \eta_{s_0}, \quad g \in G_0, \tag{5.2}$$

where $G_0 = \{g \in G : s_0 \in g\}$. In the case where such distribution exists, it is the stationary distribution of the GSMP based upon $\Sigma$ by means of $\Phi_{s_0}(\eta_s : s \in S)$.

**Remark 5.2.9** (5.2) is a set of local balance equations for the GSMP based upon $\Sigma$ by means of the exponential family in $\Phi_{s_0}(\eta_s : s \in S)$, equating the rate of transitions out of state $g$ due to the death of $s_0$ to the rate of transitions into $g$ due to the birth of $s_0$.

### 5.3 Formulation of $X(t)$ as a Queueing Process

The process $\{X(t) : t \geq 0\}$, defined in Section 5.1, can be obtained as the queueing process of an M/G/$\infty$ queue with state-dependent arrivals. Consider an infinite server queue with $L$ classes of customers (recall that $L$ represents the number of used links in the network), each class being uniquely associated with each used link in the network, and vice-versa. The successive service times of the customers of class $i$ are i.i.d. random variables with distribution function $B_i(\cdot)$. Customers arrive according to nonhomogeneous Poisson processes, whose rate at time $t$ is a function of the queue occupancy at that time, in the following manner: if $D$ is the set of customer classes present in the queue at time $t$, then the arrival process for class $i$ customers has rate $\lambda_i$ whenever the corresponding set of links $D$ (in the packet radio network) does not block link $i$, and has rate 0 otherwise. If we denote by $X(t)$ the set of customer classes present in the queue at time $t$, and restrict $X(0)$ to belong to the collection $S$ (defined in 4.1.1), then at any subsequent time $t$ we still will have $X(t) \in S$. It is easily seen that this process $\{X(t) : t \geq 0\}$ coincides with the process defined in 5.1 in terms of the link activity of the packet radio network. The correspondence between both models is made by interpreting the arrival of a class $i$ customer to the queue as the activation of link $i$ in the packet radio network, and its departure as the deactivation

of the same link. This formulation explains some of the similarities between the properties of the process describing the joint activity of the transmitters in the network, and the properties of some queueing systems, e.g., those considered in [22].

### 5.4 Formulation of $X(t)$ as a GSMP

We construct now a GSMS $\Sigma = (G, S, p)$ and a family of lifetime distributions such that the associated GSMP $\{X^*(t) : t \geq 0\}$ is equivalent to $\{X(t) : t \geq 0\}$. The states in the collection $G$ are of the form $g = \mathcal{A} \cup D$, where $D \in \mathcal{S}$ is a set of active links, and $\mathcal{A} \triangleq \{a_1, a_2, \ldots, a_L\}$ such that each $a_i$, $i \in \mathcal{L}$, is an element present in all states. The deaths of $a_i$ correspond to the occurrences of the scheduling points associated with link $i$, generating the state transitions corresponding to the activations of link $i$ if unblocked. The death of an element $j \in D$ corresponds to the deactivation of $j$. The GSMS $\Sigma = (G, S, p)$ is formally defined in the following way. Let $S \triangleq \mathcal{A} \cup \mathcal{L}$, and $G \triangleq \{g = \mathcal{A} \cup D : D \in \mathcal{S}\}$. The transition probabilities $\{p(g, s, g') : g, g' \in G, s \in g\}$ are defined by

$$p(g,s,g') = \begin{cases} 1, & \text{if } s = a_j \text{ and } j \in U(D) \text{ and } g' = g \cup \{j\} \\ & \text{or } s = a_j \text{ and } j \notin U(D) \text{ and } g' = g \\ & \text{or } s = j \text{ and } j \in D \text{ and } g' = g - \{j\}, \\ 0, & \text{otherwise.} \end{cases}$$

We associate with this GSMS a family $\varphi = \{\varphi_s : s \in S\}$ of residual lifetime distributions, such that the distribution $\varphi_{a_i}$, $i \in \mathcal{L}$, is exponential with parameter $\lambda_i$, and the distribution $\varphi_i, i \in \mathcal{L}$, is the packet length distribution for link $i$. Let $\{X^*(t) : t \geq 0\}$ denote the GSMP based upon $\Sigma$ by means of the family $\varphi$, and suppose that $X^*(t) = \mathcal{A} \cup D, D \in \mathcal{S}$. We have that $X^*(t + \Delta t) = \mathcal{A} \cup (D \cup \{i\})$, $i \in U(D)$, with probability $\lambda_i \Delta t + o(\Delta t)$, and $X^*(t + \Delta t) = \mathcal{A} \cup (D - \{j\})$ if element $j \in D$ died in the interval $(t, t + \Delta t)$. The GSMP thus obtained is equivalent to $X(t)$ in the sense that $X^*(t) = \mathcal{A} \cup X(t)$. It follows that, if we again let $\{p(D) : D \in \mathcal{S}\}$ denote the stationary distribution of $\{X(t) : t \geq 0\}$, then $p_{\mathcal{A} \cup D} = p(D)$.

**Remark 5.4.1** It is easy to see that the GSMS $\Sigma$ is irreducible (see 4.1.1). Since the intervals between the times at which new links become active are

continuous random variables, condition (C1) of Section 5.2 is satisfied for arbitrary packet length distributions. The restrictions imposed in 5.1 on these distributions ensure that (C2) is also satisfied. Given the nature of the system under study, it is intuitively plausible that (C2) be also satisfied for arbitrary packet length distributions, but we do not know of a proof of this conjecture.

### 5.5 Existence of a product form solution

Equation (4.4) does not make sense in the general case. For this case, we shall say that $\{X(t) : t \geq 0\}$ has a product form solution if its stationary distribution satisfies

$$p(D) = p(\phi) \prod_{i \in D} \frac{\lambda_i}{\mu_i}, \qquad D \in \mathcal{S}. \tag{5.3}$$

Obviously, the existence of a product form solution for $\{X(t) : t \geq 0\}$ is equivalent to (i) $\Phi_{\mathcal{L}}$-insensitivity of $\{X^*(t) : t \geq 0\}$ and (ii) the existence of a product form solution for the version of the GSMP in which all residual lifetimes are exponential; (i.e., the GSMP corresponding to the case of exponentially distributed packet lengths, which we shall refer to as the "exponential version" of the GSMP). We have the following

**Proposition 5.5.1** $\{X(t) : t \geq 0\}$ possesses a product form solution if and only if $J(D) = D$, for all $D \in \mathcal{S}$.

**Proof:** Let $\{X(t) : t \geq 0\}$ have a product form solution. In particular, the exponential version of the GSMP will have a product form solution. Proposition 4.3.5 then implies that $J(D) = D$, for all $D \in \mathcal{S}$. Conversely, let $J(D) = D$, for all $D \in \mathcal{S}$. Again from Proposition 4.3.5, the exponential version of the GSMP has a product form solution (5.3). Recall that the distribution (5.3) is a solution of the global balance equations (5.1). We shall now show that (5.3) also satisfies the local balance equations (5.2) for any $i_0 \in \mathcal{L}$. Indeed, for the system under study, the equations (5.2) take the form

$$p(D)\,\mu_{i_0} = p(D - \{i_0\})\,1_{\{i_0 \in J(D)\}}\,\lambda_{i_0}, \quad D \supseteq \{i_0\}, \quad D \in \mathcal{S} \tag{5.4}$$

or, given that $J(D) = D$,

$$p(D) = \frac{\lambda_{i_0}}{\mu_{i_0}} p(D - \{i_0\}), \quad D \supseteq \{i_0\}, \quad D \in S$$

which is indeed satisfied by (5.3). Proposition 5.2.8 then allows us to conclude that the GSMP $\{X^*(t) : t \geq 0\}$ is $\Phi_{i_0}$-insensitive for all $i_0 \in \mathcal{L}$ which, together with Proposition 5.2.5, implies that it is $\Phi_{\mathcal{L}}$-insensitive, and hence its stationary distribution is also given by (5.3). Thus $\{X(t) : t \geq 0\}$ possesses a product form solution. ∎

As a direct consequence of Propositions 4.3.4 and 5.5.1 we have (note that only the blocking properties of the protocol, and not the form of the service time distributions, are relevant for the proof of Proposition 4.3.4)

**Proposition 5.5.2** **(Criterion for the existence of a product form – general packet length distributions)** A necessary and sufficient condition for a channel access protocol, together with a given network topology and traffic requirements, in a system with general packet length distributions, to have a product form solution is that, for all pairs of *used* links $i$ and $j$, link $j$ blocks link $i$ whenever link $i$ blocks link $j$.

**Remark 5.5.3** It is possible for a given network configuration and access protocol to be insensitive with respect to the packet length distributions of a proper subset of the links of the network, and nevertheless not have a product form solution. In terms of (5.1) and (5.2), this corresponds to the solution of (5.1) satisfying (5.2) for some, but not all, links of the network. As an example, consider the network of Figure 1 operating under the I-BTMA protocol, and with nonzero traffic requirements over all links. It is easy to see that, when $i_0$ is taken to be either link 3 or 4, the corresponding system (5.4) is compatible with the solution (5.3) of (5.1), and hence insensitivity exists with respect to the packet length distributions of these

links. On the other hand, if we take $i_0$ to be any of the other links in the network, it is always possible to find states $D \in S$ such that $i_0 \not\in J(D)$, for which (5.4) then requires $p(D) = 0$. This requirement is incompatible with (5.3), thus showing that insensitivity does not exist with respect to links 1, 2, 5 and 6.

**Remark 5.5.4** In the construction of a GSMP given in Section 5.2, an element of $s \in S$ is assigned, at the times of its birth, lifetimes that can be viewed as the interarrival times in the renewal point process associated with the distribution $\varphi_s$. It can be shown that the results presented in Section 5.2 remain valid if the successive lifetimes assigned to an element $s \in S$ are obtained as the interarrival times of an *arbitrary* stationary point process with intensity $\eta_s$ [20]. This implies, in particular, that an insensitive system, as defined in Section 5.2, is also insensitive with respect to the choice of the stationary point process from which the successive lifetimes of a given active element are generated.

**Remark 5.5.5** It is easy to show that insensitivity (and hence a product form solution) does not exist with respect to the distribution of the times between occurrences of two successive scheduling points. For that purpose, let us apply Proposition 5.2.8 by taking $s_0$ to be $a_j$, $j \in \mathcal{L}$. Since the element $a_j$ is present in all states, the local balance equations (5.2) become

$$p_{\mathcal{A}\bigcup D}\,\lambda_j = \sum_{D' \in S} p_{\mathcal{A}\bigcup D'}\, p(\mathcal{A}\bigcup D', a_j, \mathcal{A}\bigcup D)\,\lambda_j, \quad D \in S.$$

By retaining only the appropriate nonzero probabilities, this equation gives

$$p_{\mathcal{A}\bigcup D}\,\lambda_j = \begin{cases} p_{\mathcal{A}\bigcup D}\,\lambda_j + p_{\mathcal{A}\bigcup(D-\{j\})}\,\lambda_j, & \text{if } j \in J(D) \\ 0, & \text{if } j \in U(D) \\ p_{\mathcal{A}\bigcup D}\,\lambda_j, & \text{if } j \in \mathcal{L} - U(D) - J(D) \end{cases} \tag{5.5}$$

Clearly this system of equations is not compatible with (5.1), since for $D$ such that $j \in U(D)$, (5.5) requires $p_{A \bigcup D} = 0$, whereas (5.1) is known to have a strictly positive solution. The conclusion then follows from Proposition 5.2.8.

**Remark 5.5.6** Although the product form (5.3) is analytically attractive, it is not satisfactory from the point of view of numerical computation. As pointed out in [12], the straightforward method of computing the normalizing factor $p(\phi)^{-1}$, given by

$$p(\phi)^{-1} = \sum_{D \in S} \prod_{i \in D} \frac{\lambda_i}{\mu_i}, \tag{5.6}$$

would involve the determination of all independent subsets of the graph with adjacency matrix $A = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1, & \text{if } i \text{ blocks } j, \text{ and } i, j \text{ are used links;} \\ 0, & \text{otherwise,} \end{cases}$$

and such determination is an NP-complete problem. We show in Appendix I that the computation of $p(\phi)$ is hard, by proving that the problem known as INDEPENDENT SET [23] can be reduced in polynomial time to a suitable version (from the point of view of computational complexity theory) of the problem of computing $p(\phi)$, thus proving that this problem is at least as hard as INDEPENDENT SET, known to be NP-complete. Hence it is not likely that a polynomial time algorithm for computing $p(\phi)$ can be found.

## 6. CONCLUSIONS

In this Report we studied the existence of product form solutions for models describing the joint activity of the transmitters in a packet radio network, under a general class of channel access protocols. For this purpose, we presented two stochastic models, corresponding to the cases of (i) exponential packet length distributions, and (ii) general packet length

distributions. The scheduling point processes associated with the links in the network were assumed to be Poisson. The stochastic process corresponding to case (i) is a Continuous Time Markov Chain, whereas that corresponding to case (ii) is a Generalized Semi-Markov Process. In case (i) we showed the existence of a product form to be equivalent to the property known as *reversibility*. We gave a criterion, valid for both cases (i) and (ii), which allows the existence of a product form solution to be easily determined from the description of the access protocol, the network topology, and the traffic pattern. This criterion does not involve the specific form of the packet length distributions. We also pointed out the equivalence between the description of the joint behavior of the transmitters in a packet radio network, and the description of the activity in a multiple server queueing system with multiple customer classes and state-dependent arrivals. In addition, we showed that a product form solution does not exist whenever any of the link scheduling point processes is not Poisson. Finally, we proved that the computation of the normalization factor of a product form solution is an NP-hard problem.

## APPENDIX I Computational complexity of normalization factor

We now show that it is possible to reduce the INDEPENDENT SET problem to the problem of computing the normalization factor (5.6). These problems are defined in terms of an undirected graph $G = (V, E)$, where $V$ is the set of vertices (or nodes), and $E$ is the set of edges (or arcs) of the graph. They are formulated as decision problems, in a form consistent with the computation model of a Turing machine. The graphs we will be interested in correspond to the description of the blocking between the links of a product form protocol, as described in Remark 5.5.6.

Given a graph $G$, we say that a set of nodes is an *independent set* if no two nodes in the set are adjacent. The two problems of interest for us are defined as follows.

**P1 (INDEPENDENT SET)** Given an undirected graph $G = (V, E)$, and an integer $K \leq |V|$, where $|V|$ denotes the cardinality of $V$, is there an independent set of vertices $V' \subseteq V$, with $|V'| \geq K$?

**P2 (SP)** Given an undirected graph $G = (V, E)$, a collection $\{\lambda_v\}_{v \in V}$ of integers, and an integer $R$, determine whether or not

$$SP(G) \triangleq \sum_{\substack{D \subseteq V \\ D\ indep.}} \prod_{i \in D} \lambda_i \geq R.$$

The input strings to these problems use the symbols "0", "1", and ",". Let $L = |V|$. The input to problem (P1) is $r_1, r_2, \ldots, r_L, K$, where $r_j$ is the $j$-th row of the adjacency matrix of $G$. The length of this input is $n_1 = (L+1)L + \lceil \log K \rceil$ (all logarithms being taken to be to the base 2), satisfying $L^2 < n_1 < L^2 + 2L$. The input to problem (P2) is $r_1, \ldots, r_L, \lambda_1, \ldots, \lambda_L, R$, of length $n_2 = L(L+1) + \sum_{i=1}^{L}(\lceil \log \lambda_i \rceil + 1) + \log R$. Note that, since $SP(G) \leq \prod_{i=1}^{L}(1+\lambda_i)$, the problem of determining the value of $SP(G)$ can be solved by doing a binary search on $\{1, \ldots, \prod_{i=1}^{L}(1+\lambda_i)\}$, taking at most $\sum_{i=1}^{L} \log(1+\lambda_i) < n_2$ calls to a solver for problem (P2).

The reduction of (P1) to (P2) in polynomial time is accomplished as follows. Given an instance of (P1), create an instance of (P2) with the same adjacency matrix, and with

$\lambda_1 = \ldots = \lambda_L = \lambda = 2^L + 1$ and $R = \lambda^K$. The length of the corresponding input string is

$$n_3 = L(L+1) + L(K+2) + K(L+1) \leq L(L+1) + L(L+2) + L(L+1) < 8L^2,$$

so that $n_3 < 8n_1$.

Suppose the largest independent set has at most $K-1$ elements. Then

$$\begin{aligned} SP(G) &= \sum_{\substack{D \subseteq V \\ D\ indep.}} \lambda^{|D|} = \sum_{j=0}^{K-1} \sum_{\substack{D \in V \\ D\ indep. \\ |D|=j}} \lambda^j \leq \sum_{j=0}^{K-1} \binom{L}{j} \lambda^j \\ &< \sum_{j=0}^{K-1} \binom{L}{\lfloor L/2 \rfloor} \lambda^j \approx \sqrt{\frac{2}{\pi L}}\, 2^L \,\frac{\lambda^K - 1}{\lambda - 1} < \lambda^K = R. \end{aligned}$$

The two last steps are justified, respectively, by Sterling's approximation, and by the fact that $\lambda \geq 2^L + 1$ implies $2^L(\lambda^K - 1)/(\lambda - 1) < \lambda^K$. Supposing now that there exists an independent subset with $K$ elements, then $SP(G) > \lambda^K = R$. Thus we found a set of parameters, $\lambda_1, \ldots, \lambda_L, R$, function of $K$ and $L$, such that the answer to an instance of (P2) is "yes" if and only if the answer to the corresponding instance of (P1) is "yes", and the translation from (P1) to (P2) can be done in polynomial time. Hence SP is at least as hard as INDEPENDENT SET.

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